A Local Error Estimator Using Finite Element Residual and Duality

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A quasistatic frictional contact formulation is derived from a dynamic contact problem using time-discretized approximation, for which a compliant interface model of the contact surface is assumed. An a posteriori error estimator is developed by a Lagrangian formulation which is recast into a set of local problems with equilibrated normal stress. Hence, local errors are calculated by solving these local element-wise problems, and can be used to obtain optimal meshes through an adaptive hp-finite element method based on a priori and a posteriori error estimates. The theory is applied to representative test problems of frictional contact, and numerical simulation results are given to support the theoretical results.

Key Words: Quasistatic, Friction, Contact Problem, Lagrangian Formulation, Equilibrated Normal Stress, A Posteriori Error, Error Estimator, Optimal Mesh, Adaptive hp-Finite Element Method

1. Introduction

In the mechanics of solids and structures, one always encounters situations in which one body is in contact with another. Nevertheless, in most analyses of solids and structures, the effects of contact are often ignored due to inherent mathematical and numerical difficulties in modeling contact. A major difficulty in treating contact problems is that the contact region or boundary is unknown a priori, which renders contact problems nonlinear. Also, in real engineering applications, contact is inherently combined with friction and a proper characterization of frictional effects on the motion of bodies in contact constitutes a highly nonlinear and much more complex problem.

Early studies of the phenomenon of friction date back to Leonardo da Vinci in the fifteenth century, Amontons in the seventeenth century and Coulomb in the eighteenth century, leading to the publication of Coulomb's laws in 1781 (Dawson, 1978). The study of contact problems as part of the theory of elasticity was started by Hertz in 1882 (Johnson, 1984). The first mathematical formulation of contact problems in elasticity was contributed by Signorini in 1933. In the last decade, new mathematical theories of frictional contact emerged with the introduction of new frictional models. Early studies of frictional contact involved the regularization of Coulomb's law or the use of nonclassical friction laws. Later, some nonclassical friction laws for contact problems involving elastic bodies were introduced by Oden and Martins (1985). Klarbing, Mikelic and Shillor (1988, 1989) formulated a variational inequality for more general contact conditions and proposed incremental and rate models which provide reasonable models for developing effective finite element approximations of elastic contact problems.

In the numerical modeling of both frictional and frictionless contact problems, a variety of questions arise. These involve such issues as what are the appropriate locations of grid points near the contact boundary, how singularities that may exist near contact boundaries can be resolved, and what choices of shape functions can best capture contact stresses. These types of numerical issues

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can be addressed using adaptive finite element methods. A key component of any adaptive process is a reliable algorithm for a priori and a posteriori error estimations. A posteriori error estimates for finite element approximations of linear elliptic problems have been popularized by Babuska and Rheinboldt and their colleagues (1978), and Kelly, Gago, and Zienkiewicz (1983) presented methods of deriving error estimates for second-order problems. Recently, a new a posteriori error estimation theory was developed by Ainsworth and Lee (1993), which provided for the construction of element-wise a posteriori error estimates by a special dual formulation in which continuity conditions at interfaces are treated as constraints.

In the present study, we consider incremental formulations of quasistatic contact problems with friction, introduce a regularization of the frictional functional, and apply the error estimators to these classes of problems. Also, numerical examples are given to support the theoretical results.

2. Formulations of Quasistatic Contact Problems

2.1 Quasistatic contact problems

We use the standard notation of Sobolev spaces and begin by considering the motion of a linearly elastic body which is assumed to be very slow in comparison with the speed of elastic waves in the body. The body is unilaterally supported by a rigid foundation and occupies a domain Ω in R^N ($N=2, 3$) which has a Lipschitz boundary Γ, and is subjected to body forces $\boldsymbol{f}$ throughout Ω and surface traction $\boldsymbol{t}$ applied to a portion Γ_F of Γ. The body is fixed along a portion Γ_D of Γ, and Γ_C denotes a candidate contact surface. The actual surface on which the body comes in contact with the foundation is not known in advance but is contained in Γ_C. The initial gap between the body and the foundation is defined by a function $g\,(g\geq 0)$. Attention is confined to infinitesimal deformations of the body, and it is assumed that the body has a linearly elastic behavior characterized by the generalized Hooke's law,

$$\sigma_{ij}(\boldsymbol{u})=E_{ijkl}u_{k,l} \text{ in } \Omega,\ 1\leq i,j,k,l\leq N \quad (1)$$

where $\boldsymbol{u}$ is a displacement field in Ω, E_{ijkl} are the usual elasticity coefficients. We denote by $\boldsymbol{n}$ the unit outward normal vector along the boundary Γ and by σ_n and $\boldsymbol{\sigma}_T$ the normal stress and tangential stress vectors on the boundary. Similarly, the displacement $\boldsymbol{u}$ on the boundary Γ is decomposed into normal and tangential components u_n and $\boldsymbol{u}_T$, respectively. The frictional contact behavior on the contact surface Γ_C is assumed to be governed by the compliant surface model studied by Oden and Martins (1985):

normal stress

$$\sigma_n(\boldsymbol{u})=-c_n(u_n-g)_+^{m_n} \quad (2)$$

tangential stress

when $u_n\leq g$

$$\boldsymbol{\sigma}_T(\boldsymbol{u})=0$$

when $u_n>g$

$$|\boldsymbol{\sigma}_T(\boldsymbol{u})|<c_T(u_n-g)_+^{m_T},\quad \dot{\boldsymbol{u}}_T=0$$

or

$$|\boldsymbol{\sigma}_T(\boldsymbol{u})|<c_T(u_n-g)_+^{m_T},$$
$$\exists\lambda\geq 0 \text{ such that } \dot{\boldsymbol{u}}_T=-\lambda\,\boldsymbol{\sigma}_T(\boldsymbol{u}) \quad (3)$$

where c_n, m_n, c_T and m_T are material interface parameters corresponding to the Coulomb's friction coefficient and $(\cdot)_+$ denotes the positive part of the argument. Assuming sufficient smoothness for all the functions and data introduced in the above relations, and equating the penetration approach to the normal distance between compliant surfaces, $a=(u_n-g)_+$, we arrive at the following system of equations and inequalities governing the displacement field $\boldsymbol{u}$, for a time interval $(0, T)$:

Linear momentum equations

$$-\rho\ddot{u}_i+\sigma_{ij,j}+f_i=0 \text{ in } \Omega\times(0, T),$$
$$1\leq i, j\leq N \quad (4)$$

Boundary conditions

$$u_i=0 \quad \text{on } \Gamma_D\times(0, T),\ 1\leq i\leq N$$
$$\sigma_{ij}n_j=t_i \text{ on } \Gamma_F\times(0, T),\ 1\leq i, j\leq N \quad (5)$$

Contact boundary conditions (2, 3)

Initial conditions

$$\boldsymbol{u}(\boldsymbol{x}, 0)=\boldsymbol{u}_0(\boldsymbol{x}) \text{ in } \Omega$$

$$\dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \dot{\boldsymbol{u}}_0(\boldsymbol{x}) \text{ in } \Omega \tag{6}$$

where ρ is the mass density of the material of which the body is composed.

Now, in order to establish a weak form of this problem, we introduce the space V of admissible displacements defined by

$$V \equiv \{\boldsymbol{v} \in [H^1(\Omega)]^N | \boldsymbol{v} = 0 \text{ on } \Gamma_D\}$$

Note that homogeneous boundary conditions are assumed on Γ_D for simplicity, but this is not essential in our theory.

The weak form of the dynamic frictional contact problem is

Find $\boldsymbol{u}(t) : (0, T) \to V$ such that

$$\langle \rho \ddot{\boldsymbol{u}}, \boldsymbol{v} - \dot{\boldsymbol{u}} \rangle + a(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + j_n(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + j_T(\boldsymbol{u}, \boldsymbol{v}) - j_T(\boldsymbol{u}, \dot{\boldsymbol{u}}) \geq F(\boldsymbol{v} - \dot{\boldsymbol{u}}) \quad \forall \boldsymbol{v} \in V \tag{7}$$

with the initial conditions

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \ \dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \dot{\boldsymbol{u}}_0(\boldsymbol{x})$$

where

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle \equiv \int_\Omega u_i v_i dx$$

$$a(\boldsymbol{u}, \boldsymbol{v}) \equiv \int_\Omega E_{ijkl} u_{i,j} v_{k,l} dx$$

$$j_n(\boldsymbol{u}, \boldsymbol{v}) \equiv \int_{\Gamma_C} c_n(\boldsymbol{s}) (u_n - g)_+^{m_n} v_n ds$$

$$j_T(\boldsymbol{u}, \boldsymbol{v}) \equiv \int_{\Gamma_C} c_T(\boldsymbol{s}) (u_n - g)_+^{m_T} |\boldsymbol{v}_T| ds$$

$$F(\boldsymbol{v}) \equiv \int_\Omega f_i v_i dx + \int_{\Gamma_F} t_i v_i ds$$

Now consider cases in which the inertia term is small enough to be neglected; for example, when the acceleration of the whole body is very small in comparison with other components in the variational inequality. Then the inertia term may be omitted and the following quasistatic formulation is obtained:

Find $\boldsymbol{u}(t) \in V$ such that

$$a(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + j_n(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + j_T(\boldsymbol{u}, \boldsymbol{v}) - j_T(\boldsymbol{u}, \dot{\boldsymbol{u}}) \geq F(\boldsymbol{v} - \dot{\boldsymbol{u}}) \quad \forall \boldsymbol{v} \in V \tag{8}$$

with the initial condition $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$.

2.2 Incremental model of the quasistatic problem

Let the time interval $[0, T]$ be divided into n successive intervals (t_l, t_{l+1}) for $l = 0, \cdots, n-1$ and $0 = t_0 < \cdots < t_n = T$. A standard finite difference approximation of displacements and velocities is introduced according to

$$\boldsymbol{u}(t_\theta) = \boldsymbol{u}(t_l) + \theta(\boldsymbol{u}(t_{l+1}) - \boldsymbol{u}(t_l)), \ 0 < \theta \leq 1$$

$$\dot{\boldsymbol{u}}(t_\theta) = \frac{1}{\Delta t}(\boldsymbol{u}(t_{l+1}) - \boldsymbol{u}(t_l))$$

where

$$t_\theta = t_l + \theta(t_{l+1} - t_l)$$

$$\Delta t = t_{l+1} - t_l$$

Here we assume equilibrium at $t = t_\theta$ and introduce new notations as follows:

$$\boldsymbol{u}^l \equiv \boldsymbol{u}(t_l)$$

$$\boldsymbol{w} \equiv \theta(\boldsymbol{u}^{l+1} - \boldsymbol{u}^l)$$

$$\boldsymbol{v} \equiv \theta \cdot \Delta t \cdot \boldsymbol{v}$$

$$g^l \equiv g - u_n^l$$

$$j_n^l(\boldsymbol{u}, \boldsymbol{v}) \equiv \int_{\Gamma_c} c_n(\boldsymbol{s}) (u_n - g^l)_+^{m_n} v_n ds$$

$$j_T^l(\boldsymbol{u}, \boldsymbol{v}) \equiv \int_{\Gamma_c} c_T(\boldsymbol{s}) (u_n - g^l)_+^{m_T} |\boldsymbol{v}_T| ds$$

$$F^l(\boldsymbol{v}) \equiv \int_\Omega f_i(\boldsymbol{x}, t_\theta) v_i dx + \int_{\Gamma_F} t_i(\boldsymbol{s}, t_\theta) v_i ds - a(\boldsymbol{u}^l, \boldsymbol{v})$$

Then we obtain:

Find $\boldsymbol{w} \in V$ such that

$$a(\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}) + j_n^l(\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}) + j_T^l(\boldsymbol{w}, \boldsymbol{v}) - j_T^l(\boldsymbol{w}, \boldsymbol{w}) \geq F^l(\boldsymbol{v} - \boldsymbol{w}) \quad \forall \boldsymbol{v} \in V \tag{9}$$

Here $\boldsymbol{u}^l$ is known and treated as data at each time. Thus, at each time step, we obtain a time independent problem which is similar to static contact problems.

2.3 A minimization problem

When friction is ignored, the variational inequality (9) reduces to the following nonlinear equation:

Find $\boldsymbol{w} \in V$ such that

$$a(\boldsymbol{w}, \boldsymbol{v}) + j_n^l(\boldsymbol{w}, \boldsymbol{v}) = F^l(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V \tag{10}$$

In order to formulate a minimization problem, we define a potential functional J_n^l such that

$$J_n^l(\boldsymbol{v}) \equiv \frac{1}{m_n + 1} \int_{\Gamma_c} c_n(\boldsymbol{s}) (v_n - g^l)_+^{m_n+1} ds$$

which is convex and differentiable. Then a minimization problem which is equivalent to (10) can be formulated as follows:

Find $\boldsymbol{w} \in V$ such that

$$T^l(\boldsymbol{w}) = \inf_{v \in V} T^l(\boldsymbol{v})$$

where

$$T^l(\boldsymbol{v}) \equiv \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) + J_n^l(\boldsymbol{v}) - F^l(\boldsymbol{v}) \quad (11)$$

3. Finite Element Approximation

Suppose that the domain Ω is exactly covered by finite elements Ω_K, i.e.,

$$\bar{\Omega} = \bigcup_K \bar{\Omega}_K$$

Let $V^h \subset V$ be the finite element space defined by

$$V^h \equiv [V^{hp}(\Omega)]^N \cap V$$

where V^{hp} is the space of piecewise continuous polynomials. The discrete problems corresponding to (10) and (11) are characterized as follows: Find $\boldsymbol{w}^h \in V^h$ such that

$$a(\boldsymbol{w}^h, \boldsymbol{v}^h) + j_n^l(\boldsymbol{w}^h, \boldsymbol{v}^h) = F^l(\boldsymbol{v}^h) \quad \forall \ \boldsymbol{v}^h \in V^h \quad (12)$$

and
Find $\boldsymbol{w}^h \in V^h$ such that

$$T^l(\boldsymbol{w}^h) = \inf_{v^h \in V^h} T^l(\boldsymbol{v}^h) \quad (13)$$

4. An *a Posteriori* Error Estimation

4.1 An upper bound of the error

First, we apply Ainsworth-Lee *a posteriori* error estimators to those nonlinear equations which characterize contact problems without friction. Later, validity of the error estimators will be discussed in the presence of friction. Attention is confined to two-dimensional problems for simplicity.

Let $\boldsymbol{w}$ and $\boldsymbol{w}^h$ be the solutions of (10) and (12), respectively. The finite element approximation error $\boldsymbol{e}$ and the energy norm are defined as

$$\boldsymbol{e} \in V, \ \boldsymbol{e} \equiv \boldsymbol{w} - \boldsymbol{w}^h$$
$$\|\boldsymbol{v}\|_E \equiv \sqrt{a(\boldsymbol{v}, \boldsymbol{v})}$$

The difference of the potential functions $T^l(\boldsymbol{w})$ and $T^l(\boldsymbol{w}^h)$ can be reduced to the following:

$$T^l(\boldsymbol{w}) - T^l(\boldsymbol{w}^h) = -\frac{1}{2} a(\boldsymbol{e}, \boldsymbol{e}) + J_n^l(\boldsymbol{w}) - J_n^l(\boldsymbol{w}^h) - j_n^l(\boldsymbol{w}, \boldsymbol{w}) + j_n^l(\boldsymbol{w}, \boldsymbol{w}^h) \quad (14)$$

From the convexity of J_n^l and j_n^l, we have

$$\|\boldsymbol{e}\|_E^2 = a(\boldsymbol{e}, \boldsymbol{e}) \leq -2\{T^l(\boldsymbol{w}) - T^l(\boldsymbol{w}^h)\} = -2 \inf_{v \in V} \{T^l(\boldsymbol{v}) - T^l(\boldsymbol{w}^h)\} \quad (15)$$

which states that the energy norm of error is bounded by the difference of the potential functionals.

4.2 Localization of error estimator

Now it is necessary to localize the global upper bound. First, local solution spaces V_K and V_{loc} are defined in a finite element Ω_K as

$$V_K \equiv \{\boldsymbol{v} \in [H^1(\Omega_K)]^2 | \boldsymbol{v} = 0 \text{ on } \partial\Omega_K \cap \Gamma_D\}$$

and

$$V_{loc} \equiv \prod_K V_K$$

so that $V \subset V_{loc}$. With these local spaces V_K, the nonlinear terms are restricted to a finite element Ω_K as follows:

$$a_K(\boldsymbol{u}_K, \boldsymbol{v}_K) \equiv \int_{\Omega_K} E_{ijkl} u_{Ki,j} v_{Kk,l} dx$$
$$J_{nK}^l(\boldsymbol{v}_K) \equiv \frac{1}{m_n + 1} \int_{\partial\Omega_K \cap \Gamma_C} c_n(\boldsymbol{s}) (v_{Kn} - g^l)_+^{m_n+1} ds$$
$$F_K^l(\boldsymbol{v}_K) \equiv \int_{\Omega_K} f_i v_{Ki} dx + \int_{\partial\Omega_K \cap \Gamma_F} t_i v_{Ki} ds$$
$$T_K^l(\boldsymbol{v}_K) \equiv \frac{1}{2} a_K(\boldsymbol{v}_K, \boldsymbol{v}_K) + J_{nK}^l(\boldsymbol{v}_K) - F_K^l(\boldsymbol{v}_K)$$

Also, an averaging function $\alpha_{KL}^{(i)}(\boldsymbol{s})$ is introduced on the interelement boundaries Γ_{KL} shared by elements Ω_K and Ω_L such as (see Lee and Oden, 1994)

$$\alpha_{KL}^{(i)}(\boldsymbol{s}) + \alpha_{LK}^{(i)}(\boldsymbol{s}) = 1, \quad i = 1, 2, \quad \boldsymbol{s} \in \Gamma_{KL}$$

where i denotes a component in i-direction. In terms of this averaging function, the averaged normal stress and the jump on Γ_{KL} are defined as *averaged normal stress*:

$$\langle \sigma_{ij}(\boldsymbol{w}^h) n_{Kj} \rangle_{1-\alpha} \equiv \alpha_{KL}^{(i)}(\boldsymbol{s}) \sigma_{ij}(\boldsymbol{w}_K^h) n_{Kj} + \alpha_{LK}^{(i)}(\boldsymbol{s}) \sigma_{ij}(\boldsymbol{w}_L^h) n_{Kj}$$

jump:

$$\|\boldsymbol{v}\|_i \equiv \begin{cases} v_{Ki} - v_{Li}, & \text{if } K > L \\ v_{Li} - v_{Ki}, & \text{if } K < L \end{cases}$$

where $\boldsymbol{n}_K$ is an outward unit normal vector of $\partial\Omega_K$ and $\boldsymbol{w}_K^h$ is a restriction of $\boldsymbol{w}^h$ to Ω_K.

We next extend T^l to the space V_{loc} and consider the difference of functionals,

$$I^l(\boldsymbol{v}) \equiv T^l(\boldsymbol{v}) - T^l(\boldsymbol{w}^h),\ \boldsymbol{v} \in V_{loc}$$

Then $I^l(\boldsymbol{v})$ can be recast into the following:

$$I^l(\boldsymbol{v}) = \sum_K \{T_K^l(\boldsymbol{v}) - T_K^l(\boldsymbol{w}^h) - \int_{\partial\Omega_K\setminus\partial\Omega} \langle\sigma_{ij}(\boldsymbol{w}^h)\, n_{Kj}\rangle_{1-\alpha}(v_i - w_i^h)\, ds\} + \sum_{\Gamma_{KL}} \int_{\Gamma_{KL}} \|\boldsymbol{v} - \boldsymbol{w}^h\|_i \langle\sigma_{ij}(\boldsymbol{w}^h)\, n_j\rangle_{1-\alpha} ds \quad (16)$$

where $\boldsymbol{n}$ denotes the unit normal vector defined on each interelement edge which points the element with larger element number.

By penalizing the interelement jump $\|\boldsymbol{v} - \boldsymbol{w}^h\|$ by a Lagrange multiplier $\boldsymbol{\mu}$, a Lagrange functional L^l is introduced such that

$$L^l(\boldsymbol{v}, \boldsymbol{\mu}) \equiv I^l(\boldsymbol{v}) - \boldsymbol{\mu}(\|\boldsymbol{v} - \boldsymbol{w}^h\|)$$

Now, using standard functional analysis, it is not difficult to obtain the following inequalities:

$$\begin{aligned} -\frac{1}{2}\|\boldsymbol{e}\|_E^2 \geq I^l(\boldsymbol{w}) &= \inf_{\boldsymbol{v}\in V_{loc}} \sup_{\boldsymbol{\mu}\in M} L^l(\boldsymbol{v}, \boldsymbol{\mu}) \\ &\geq \sup_{\boldsymbol{\mu}\in M} \inf_{\boldsymbol{v}\in V_{loc}} L^l(\boldsymbol{v}, \boldsymbol{\mu}) \\ &\geq \inf_{\boldsymbol{v}\in V_{loc}} L^l(\boldsymbol{v}, \hat{\boldsymbol{\mu}}),\quad \forall \hat{\boldsymbol{\mu}} \in \boldsymbol{M} \quad (17) \end{aligned}$$

where $\boldsymbol{M}$ denotes the space of Lagrange multipliers.

With a special choice of a Lagrange multiplier $\hat{\boldsymbol{\mu}}$, an upper bound of the error is given as follows:

$$\|\boldsymbol{e}\|_E^2 \leq -2\sum_K \inf_{\boldsymbol{v}_K\in V_K} \{T_K^l(\boldsymbol{v}_K) - T_K^l(\boldsymbol{w}^h) - \int_{\partial\Omega_K\setminus\partial\Omega} \langle\sigma_{ij}(\boldsymbol{w}^h)\, n_{Kj}\rangle_{1-\alpha} v_{Ki} ds + \int_{\partial\Omega_K\setminus\partial\Omega} \langle\sigma_{ij}(\boldsymbol{w}^h)\, n_{Kj}\rangle_{1-\alpha} w_i^h ds\} \quad (18)$$

where $\hat{\boldsymbol{\mu}}$ is chosen as

$$\hat{\boldsymbol{\mu}}(\|\boldsymbol{v} - \boldsymbol{w}^h\|) = \sum_{\Gamma_{KL}} \int_{\Gamma_{KL}} \|\boldsymbol{v} - \boldsymbol{w}^h\|_i \langle\sigma_{ij}(\boldsymbol{w}^h)\, n_j\rangle_{1-\alpha} ds \quad (19)$$

Hence, in order to obtain an upper bound of the approximation error, we only have to solve the following element-wise problem:

Find $\boldsymbol{w}_K \in V_K$ such that $\forall \boldsymbol{v}_K \in V_K$

$$a_K(\boldsymbol{w}_K, \boldsymbol{v}_K) + j_{nK}^l(\boldsymbol{w}_K, \boldsymbol{v}_K) - F_K^l(\boldsymbol{v}_K) - \int_{\partial\Omega_K\setminus\partial\Omega} \langle\sigma_{ij}(\boldsymbol{w}^h)\, n_{Kj}\rangle_{1-\alpha} v_{Ki} ds = 0 \quad (20)$$

This boundary value problem characterizes the situation in which the original boundary condition is applied to any of the element boundaries and the equilibrated normal stress is exerted on the interelement boundaries.

Once a solution $\hat{\boldsymbol{w}}_K$ is obtained for (20), an upper bound to the error can be calculated according to:

$$\|\boldsymbol{e}\|_E^2 \leq -2\sum_K \{\bar{T}_K^l(\hat{\boldsymbol{w}}_K) - \bar{T}_K^l(\boldsymbol{w}^h)\} \quad (21)$$

where

$$\bar{T}_K^l(\boldsymbol{v}_K) = T_K^l(\boldsymbol{v}_K) - \int_{\partial\Omega_K\setminus\partial\Omega} \langle\sigma_{ij}(\boldsymbol{w}^h)\, n_{Kj}\rangle_{1-\alpha} v_{Ki} ds$$

Or, using (14),

$$\begin{aligned} &\bar{T}_K^l(\hat{\boldsymbol{w}}_K) - \bar{T}_K^l(\boldsymbol{w}^h) \\ &= -\frac{1}{2} a_K(\hat{\boldsymbol{w}}_K - \boldsymbol{w}^h,\ \hat{\boldsymbol{w}}_K - \boldsymbol{w}^h) \\ &\quad + J_K^l(\hat{\boldsymbol{w}}_K) - J_K^l(\boldsymbol{w}^h) - j_{nK}^l(\hat{\boldsymbol{w}}_K,\ \hat{\boldsymbol{w}}_K - \boldsymbol{w}^h) \end{aligned}$$

we obtain

$$\begin{aligned} \|\boldsymbol{e}\|_E^2 \leq\ & a_K(\hat{\boldsymbol{w}}_K - \boldsymbol{w}^h,\ \hat{\boldsymbol{w}}_K - \boldsymbol{w}^h) \\ & - 2[J_K^l(\hat{\boldsymbol{w}}_K) - J_K^l(\boldsymbol{w}^h) \\ & - j_{nK}^l(\hat{\boldsymbol{w}}_K,\ \hat{\boldsymbol{w}}_K - \boldsymbol{w}^h)] \end{aligned} \quad (22)$$

Moreover, we can formulate discretized problems by introducing the local solution space V_K^h as:

$$V_K^h \equiv [V^{hp}(\Omega_K)]^N \cap V_K$$

Then the discrete problem corresponding to (20) is characterized as follows:

Find $\boldsymbol{w}_K^h \in V_K^h$ such that $\forall \boldsymbol{v}_K^h \in V_K^h$

$$a_K(\boldsymbol{w}_K^h,\ \boldsymbol{v}_K^h) + j_{nK}^l(\boldsymbol{w}_K^h,\ \boldsymbol{v}_K^h) - F_K^l(\boldsymbol{v}_K^h) - \int_{\partial\Omega_K\setminus\partial\Omega} \langle\sigma_{ij}(\boldsymbol{w}^h)\, n_{Kj}\rangle_{1-\alpha} v_{Ki}^h ds = 0 \quad (23)$$

or equivalently,

Find $\boldsymbol{w}_K^h \in V_K^h$ such that

$$\bar{T}_K^l(\boldsymbol{w}_K^h) = \inf_{\boldsymbol{v}_K\in V_K^h} \bar{T}_K^l(\boldsymbol{v}^h) \quad (24)$$

Here note that since $\hat{\boldsymbol{w}}_K^h$ is expected to be close to $\hat{\boldsymbol{w}}_K$, it is reasonable to use better approximations for $\hat{\boldsymbol{w}}_K^h$ than those for $\boldsymbol{w}^h$. Either the p- or the h-method can be used to get better approximations for this purpose.

4.3 Contact problems with friction

With friction the minimization problem cannot be formulated, because potential functions are not available as in the frictionless case. From the mathematical point of view, the frictional contact problems are path (history) dependent while frictionless problems are not. Nevertheless we can still use the Ainsworth-Lee error estimators without the frictional boundary term, and include the frictional terms only in the residual calculation procedure. The error estimator is expected to still capture the errors which arise from the inside of the domain and normal contact boundary, as is validated by numerical experiments in the following section.

5. Numerical Experiments

Methods and numerical techniques for resolving the local problems for $\hat{w}_K^h$ is well explained in (Lee and Oden, 1994). Here example problems are solved and numerical results are given to support the theory. We consider a two-dimensional plane strain problem involving an infinitely long linearly elastic cylinder in contact with a rigid flat foundation. From the symmetry of the problem, only half of the body is considered and the configuration of the problem is shown in Fig. 1. Young's modulus and Poisson's ratio are taken as $E=1.4\times10^3(10^5\text{kg/cm/sec}^2)$ and $\nu=0.25$. Contact parameters are taken as follows: $c_n=1.0\times10^3(10^5\text{kg/cm}^3/\text{sec}^2)$, $c_T=0.3\times10^3(10^5\text{kg/cm}^3/\text{sec}^2)$, $m_n=m_T=2$. Body forces are neglected, and two different types of uniform pressure are applied on the upper boundary of the body as shown in Fig. 2. The time step is taken as 1 (sec) and the range of time is from 0 to 10 (sec). The initial conditions used are the following: the cylinder rests on the rigid flat foundation without any external force, and it is in an equilibrium position as $\boldsymbol{w}(\boldsymbol{x}, 0)=0$ and $\dot{\boldsymbol{w}}(\boldsymbol{x}, 0)=0$

The problem is solved using quasistatic theory, and a finite element analysis of the problems is performed using quadrilateral elements with shape functions of uniform order $p=2$. The regularization parameter is taken to be 10^{-1} (cm), and the Newton-Raphson method is used to solve the resulting nonlinear problems at each time step. Since exact solutions are not available, the quality of the error estimates is judged by calculating the error based on the reference solutions, presumed to be very accurate, which are obtained by the same finite elements with element order $p=8$. The errors are estimated with and without normal stress (flux) equilibration, which means that the equilibration parameter, $\alpha_{KL}^{(i)}$, is taken as 0.5.

In order to compare the estimated errors with the exact errors, the global and local effectivity

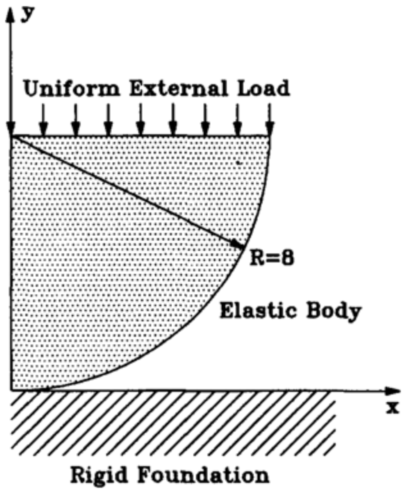

Fig. 1 Configuration of the problem: a cylindrical elastic body resting on a rigid flat foundation.

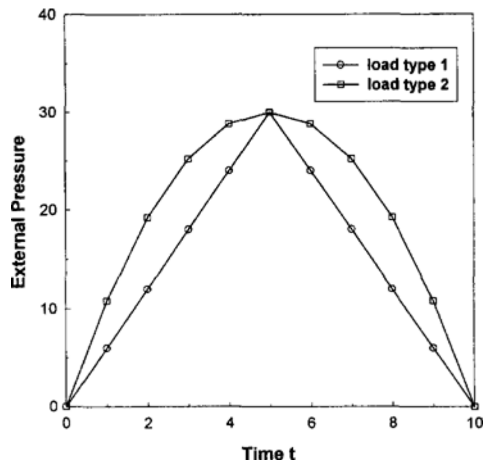

Fig. 2 Time history of load type 1 and 2.

indices are defined as the ratios of the estimated global and local errors to the exact global and local errors, respectively. These effectivity indices are to testify the quality of the error estimators, and the values of indices close to one reveal that the estimated errors are close to the exact errors, which means that the error estimators are accurate.

Tables 1 and 2 show the global errors and effectivity indices in each time step which are

Table 1 Global errors and effectivity indices along time steps (load type 1).

Time step	Error			Effectivity index	
	True	With equilibrium	Without equilibrium	With equilibration	Without equilibration
1	0.1379361	0.1841161	0.1828515	1.33	1.33
2	0.2194921	0.2817038	0.2879182	1.28	1.31
3	0.2614171	0.3257678	0.3575906	1.25	1.37
4	0.3093635	0.4032462	0.4400417	1.30	1.42
5	0.4030606	0.5433496	0.5720164	1.35	1. 42
6	0.3187154	0.4131368	0.4549692	1.30	1.43
7	0.2743482	0.3471121	0.3782150	1.27	1.38
8	0.2348655	0.2860148	0.3050285	1.22	1.30
9	0.1531524	0.1781268	0.1968649	1.16	1.29
10	0	0	0	0.31	0.31

Table 2 Global errors and effectivity indices along time steps (load type 2).

Time step	Error			Effectivity index	
	True	With equilibrium	Without equilibrium	With equilibration	Without equilibration
1	0.1934476	0.2459342	0.1828515	1.27	1.38
2	0.2568225	0.3174395	0.3502144	1.24	1.36
3	0.3105912	0.4095945	0.4437929	1.32	1.43
4	0.3684494	0.4911055	0.5282425	1.33	1.43
5	0.4096880	0.5465053	0.5791954	1.33	1.41
6	0.3773838	0.4928263	0.5386519	1.31	1.43
7	0.3222945	0.4202081	0.4625744	1.30	1.44
8	0.2731543	0.3434459	0.3760057	1.26	1.38
9	0.2123726	0.2520066	0.2815549	1.19	1.33
10	0	0	0	0.32	0.32

calculated by two different methods. From the effectivity indices, the estimated errors are observed to be close to the true errors in every time step, and the equilibrated errors are closer to the true errors than the averaged ones. Since no external force is present after time step 10, a zero solution is obtained. Hence the effectivity indices are expected to be inaccurate, as indicated in Tables 1 and 2.

A typical time step is chosen at $t=3$, and Figs. 3, 4 and 5 show the distribution of local true and estimated errors with and without equilibration of normal stress for load type 1. Also, distribution of local effectivity indices are shown in Figs. 6 and 7. As stated earlier, the theory is restricted to global results and frictionless cases. But, in Figs. 3~7, the local results are observed to be also as close to the true errors as the global results. In other words, the error estimation is more accurate with equilibration in local elementwise errors as well as global errors. Figs. 8~12 show the results for load type 2 which lead to the same conclusion.

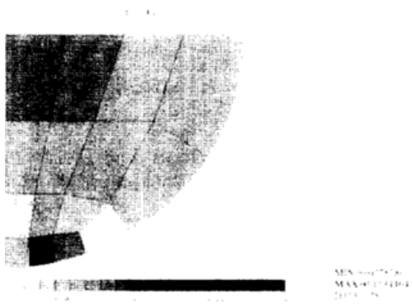

Fig. 3 Distribution of local true errors at $t=3$ (load type 1).

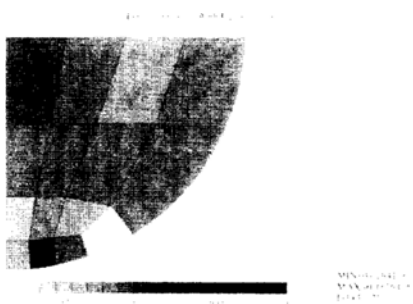

Fig. 4 Distribution of local errors estimated with equilibration at $t=3$ (load type 1).

6. Conclusions

In this paper, a quasistatic frictional contact formulation is derived from a dynamic contact problem, and an a posteriori error estimator is developed by a Lagrangian formulation and localization of minimization problems. The the-

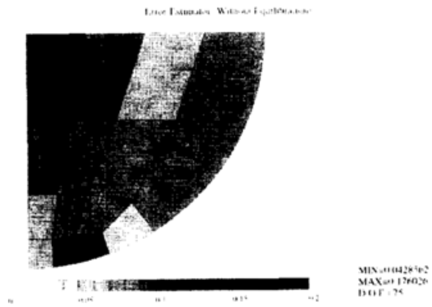

Fig. 5 Distribution of local errors estimated without equilibration at $t=3$ (load type 1).

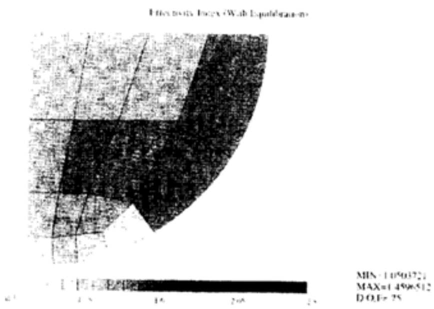

Fig. 6 Distribution of local effectivity indices estimated with equilibration at $t=3$ (load type 1).

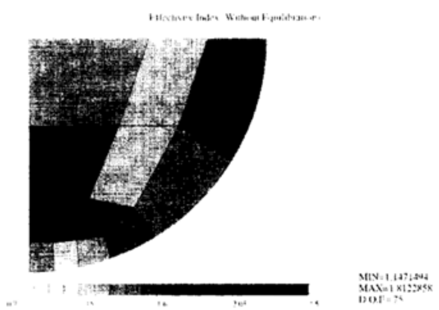

Fig. 7 Distribution of local effectivity indices estimated without equilibration at $t=3$ (load type 1).

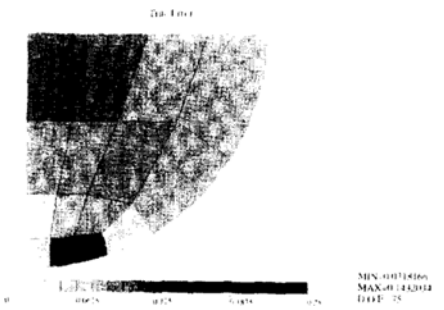

Fig. 8 Distribution of local true errors at $t=3$ (load type 2).

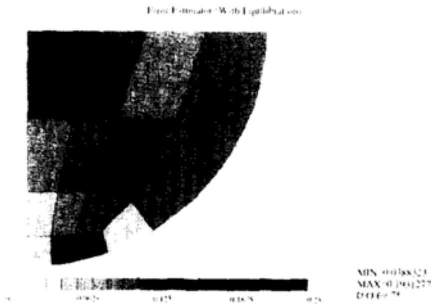

Fig. 9 Distribution of local errors estimated with equilibration at $t=3$ (load type 2).

Fig. 10 Distribution of local errors estimated without equilibration at $t=3$ (load type 2).

ory is verified by numerical experiments, the results of which lead to the following conclusions:

(1) The error estimator is based on the global upper bound of the minimization problems which holds for only global errors, but the local errors are also accurately captured by the error estimators.

(2) Even though the error estimator is devel-

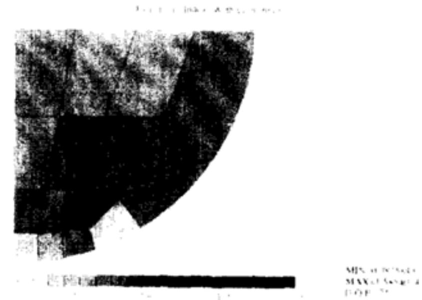

Fig. 11 Distribution of local effectivity indices estimated with equilibration at $t=3$ (load type 2).

Fig. 12 Distribution of local effectivity indices estimated without equilibration at $t=3$ (load type 2).

oped only for frictionless contact problems, the effective indices show that the error estimator can accurately capture the errors even in frictional cases.

(3) The equilibrated errors are generally closer to the true errors than those obtained by simple averaging, and this is found to be true for both the general distribution of errors over the mesh and the local effectivity indices.

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